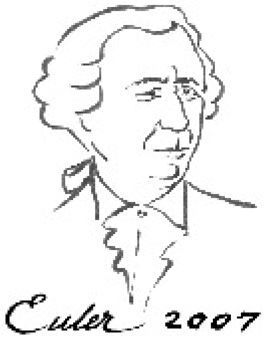

... and what does Euler say?

A little anthology of quotations
compiled for the special exhibition
"Experiencing Mathematics"
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Selection and translation:
Martin Mattmüller
Euler-Archiv Basel

On the cognitive value of “higher” mathematics

These days nearly nobody doubts any more the great utility of mathematics, since the various disciplines and arts necessary for everyday life cannot be treated without its help. However, many will hold that this praise belongs just to the lower parts of this science, its elements so to speak, whereas “higher” mathematics is denied any usefulness. To those who think this way, higher mathematics seems to be like a spider’s web which is unfit for any use because of its great delicateness. But since all mathematics is dealing with the investigation of unknown quantities, it develops methods for this – opening, as it were, pathways leading towards truth – and draws forth the most hidden truths into full light. Thus it sharpens the mind’s faculties and extends our understanding: two aims which are certainly worth every effort. For since every truth is not just praiseworthy by itself, but also for the eminent connection by which all truths are linked, it can never be useless, even if the utility is not at once understood. So the reproach that higher mathematics is digging too deep in its search for truth should be rather considered a reason for praising this science than for blaming it.

Quanquam nunc quidem summa matheseos utilitas a nemine in dubium vocari solet, propterea quod variae disciplinae et artes in vita communi necessariae sine eius cognitione tractari nequeunt, haec tamen laus a plerisque inferioribus tantum istius scientiae partibus et tanquam elementis ita propria esse putatur, ut eam partem, quae ob excellentiam sublimior vocari solet, omni usu atque utilitate carere arbitrentur. His scilicet, qui ita sentiunt, mathesis sublimior telae araneae similis videtur, quae ob nimiam subtilitatem omni utilitate destituatur. Cum autem universa mathesis in investigatione quantitatum incognitarum versetur atque in hunc finem vel methodos et quasi vias ad veritatem ducentes patefaciat, vel ipsas veritates maxime reconditas eruat atque in lucem protrahat, quorum altero vis ingenii acuitur, altero cognitio nostra amplificatur: in neutro certe nimium operae collocari potest. Cum enim veritas non solum ipsa per se sit laudabilis, sed etiam ob summum nexum, quo cunctae veritates inter se cohaerent, utilitate vacare nequeat, etiamsi non statim usus perspiciatur, obiectio illa, qua mathesis sublimior nimis profunde in investigatione veritatis penetrare arguitur, in laudationem potius quam vituperium scientiae vertitur.

E 790
Commentatio de Matheseos sublimioris utilitate
(written ca.1741 for Frederick II. of Prussia)
Journal für die reine und angewandte Mathematik 35 (1847)
Opera III 2, p.392

Reverence for Creation

Now I can explain to Your Highness in what way the faculty of sight works in the eyes of humans and animals; without doubt this is the most wondrous thing which the human mind has ever been called upon to consider. Although we are far from perfectly understanding it, the little we know is sufficient to convince us of the omnipotence and the infinite wisdom of the Creator ... How can those self-willed strong spirits who dismiss everything that their limited reason does not comprehend not be moved by this contemplation!

[after the half-blind old scientist has explained to his young pupil the structural details of the human eye]

Jetzo bin ich in der Lage, Euer Hoheit zu erklären, auf was für eine Art das Sehen in den Augen der Menschen und der Thiere vorgehe, welches ohne Zweifel die wunderbarste Sache ist, zu deren Erkenntniss der menschliche Verstand nur hat kommen können. Ob wir gleich bey weitem es nicht vollkommen kennen, so ist doch das wenige, was wir wissen, hinlänglich, uns von der Allmacht und der unendlichen Weisheit des Schöpfers zu überzeugen ... Wie sollten doch die starken Geister, die alles verwerfen, was ihr eingeschränkter Verstand nicht begreift, durch diese Betrachtung gerührt werden!

E 343
Briefe an eine deutsche Prinzessin, letter XI
in the German translation of 1769
quoted after the reprint, Braunschweig 1986, p.47

How do I find out where I am?

Indeed, when a traveller after a long journey reaches some location – on shore or on sea –, nothing will be of more interest for him than to learn at which spot of the earth he actually is, whether he is near some known place or not, and what direction he should take to arrive there. No doubt the only means to help this traveller out of his quandary will be to disclose to him the latitude and longitude of the place where he is located: but what means should he use to discover these?

[In what follows, Euler proposes six methods for determining one's location; since the techniques for the measurement of time at sea and those involving magnetic inclination are still insufficiently developed, he decides in favour of the methods based on astronomical observations, particularly on the position of the Moon.]

En effet si un homme après un long voyage arrive à un endroit, soit sur terre, soit sur mer, rien ne sauroit etre plus interessant pour lui, que d'apprendre en quel lieu de la terre il se trouve alors; s'il est proche de quelque pays connu, ou non ? et quel chemin il faut prendre pour y arriver ? Le seul moyen de tirer cet homme de son embarras sera sans doute de lui decouvrir la latitude et la longitude du lieu ou il se trouve: mais de quel moyen doit-il se servir pour parvenir à cette decouverte?

E 417
Lettres à une princesse d'Allemagne, letter CLX
St-Pétersbourg 1772
Opera III 12, p.73

Model building in natural science

Although it is not granted to us to penetrate the innermost secrets of Nature and thus to discern the true causes of phenomena, it may all the same happen that some hypothesis we form suffices for explaining several phenomena just as well as if we had indeed perceived their true cause. In this way almost all celestial motions are determined these days with the greatest success by the hypothesis of universal attraction, even though this hypothesis itself must definitely be banished from Physics. It is therefore possible that in a similar way some hypothesis could be devised which should suffice to explain all the phenomena of air and the atmosphere. ... I had thus conceived in my mind the nature of air as if it were composed of countless very small bubbles or globules, each enclosed by a thin water film; inside these the airy matter properly speaking would be rotating in a very fast circular motion, whose centrifugal force apparently gives rise to the elasticity of the air.

[In what follows, Euler deduces the relation between density, pressure, temperature, and humidity in the atmosphere from this concept – developing one of the first approaches that tries to explain macroscopic effects by a microscopic model.]

Quanquam nobis in intima naturae mysteria penetrare, indeque veras caussas Phaenomenorum agnoscere neutiquam est concessum: tamen evenire potest, ut hypothesis quaedam ficta pluribus phaenomenis explicandis aeque satisfaciat, ac si vera caussa nobis esset perspecta, quemadmodum felicissimo successu omnes fere motus coelestes ex hypothesi attractionis universalis determinari solent, etiamsi haec ipsa hypothesis ex Physica prorsus sit profliganda. Quam ob rem fortasse simili modo quaepiam hypothesis excogitari poterit, quae omnibus Phaenomenis aëris et atmosphaerae explicandis sufficiat. ... Naturam aëris autem ita animo conceperam, quasi ex innumerabilibus minimis bullulis seu sphaerulis esset compositus, quae singulae cuticula tenuissima aquosa circumdarentur, intra quas propria aëris materia motu rapidissimo in gyrum circumagatur, in cuius vi centrifuga elasticitas aëris produci erat visa.

E 527
Coniectura circa naturam aëris, pro explicandis phaenomenis in atmosphaera observatis
Acta Academiae Scientiarum Petropolitanae 1779/I (1782)
Opera II 31, p.307-308

Mathematics without measurement and calculation

Besides that part of geometry which deals with quantities and has always been studiously cultivated, there is another branch which is still virtually unknown; this was first mentioned by Leibniz, who called it the “geometry of position” [*geometric situs*]. It is concerned, he says, just with the determination of position and the investigation of its properties, and for these goals no quantities are to be considered nor any calculations with quantities used. However it has not yet been clearly defined what problems belong to this geometry of position and what method must be used to solve them. Hence, when a certain problem was recently mentioned to me which seemed to belong to geometry but did not ask for the determination of any quantities and did not admit a solution by the calculation of quantities either, I did not doubt that this belongs to the geometry of position – all the more since in its solution only positions are considered and no use is made of calculations. I propose therefore to explain here the method which I invented for the solution of this kind of problems, as a specimen of the geometry of position.

[The problem that follows – the search for a walking tour on the seven bridges of Königsberg – and Euler’s solution are considered to be at the origin of graph theory, an important part of the field now called topology.]

Praeter illam geometriae partem, quae circa quantitates versatur et omni tempore summo studio est exculta, alterius partis etiamnum admodum ignotae primus mentionem fecit Leibnitzius, quam Geometriam situs vocavit. Ista pars ab ipso in solo situ determinando situsque proprietatibus eruendis occupata esse statuitur; in quo negotio neque ad quantitates respiciendum neque calculo quantitatum utendum sit. Cuiusmodi autem problemata ad hanc situs geometriam pertineant et quali methodo in iis resolvendis uti oporteat, non satis est definitum. Quamobrem, cum nuper problematis cuiusdam mentio esset facta, quod quidem ad geometriam pertinere videbatur, at ita erat comparatum, ut neque determinationem quantitatum requireret neque solutionem calculi quantitatum ope admitteret, id ad geometriam situs referre haud dubitavi, praesertim quod in eius solutione solus situs in considerationem veniat, calculus vero nullius prorsus sit usus. Methodum ergo meam, quam ad huius generis problemata solvenda inveni, tanquam specimen Geometriae situs hic exponere constitui.

E 53
Solutio problematis ad Geometriam Situs pertinentis
Commentarii Academiae Scientiarum Petropolitanae 8, 1736 (1741)
Opera I 7, p.1

Pleasure and insight – for example in music

It is an important and curious question why beautiful music stimulates a sense of pleasure within us. Scholars still disagree about the reason: some suppose that it is by pure caprice, and that the pleasure induced by music is not at all based on reasons, considering that the same music can gratify some persons and displease others. ... Other scholars say that the pleasure one takes in listening to beautiful music is based on the perception of the order reigning within it. ... He who hears some music and understands, by the judgment of his ears, all the proportions on which both harmony and rhythm are based, surely has the most perfect knowledge of that music which is possible; whereas some other person who perceives these proportions only partly or not at all, understands nothing or has an imperfect knowledge. ... So it can be said that pleasure demands an insight which should not be too easy but requires some effort; the perception should as it were cost us something.

C'est une question aussi importante que curieuse, pourquoi une belle musique excite en nous le sentiment du plaisir? Les savans sont bien partagés là-dessus. Il y en a qui pretendent, que c'est une pure bizarrerie, et que le plaisir que cause la musique, n'est fondé sur aucune raison, vu que la même musique peut être goûtée par quelques uns, et déplaire à d'autres. ... D'autres disent que le plaisir qu'on sent en entendant une belle musique, consiste dans la perception de l'ordre qui y regne. ... Qui entend une musique, et qui comprend, par le jugement de ses oreilles, toutes les proportions sur lesquelles tant l'harmonie que la mesure est fondée, il est certain qu'il a la plus parfaite connoissance de cette musique qui soit possible; pendant qu'un autre qui n'apperçoit ces proportions qu'en partie, ou point du tout, n'y comprend rien, ou en a une connoissance imparfaite. ... On dit donc que le plaisir demande une connoissance qui ne soit pas trop facile, mais qui exige quelque peine; il faut pour ainsi dire, que cette connoissance nous coute quelque chose.

E 343
Lettres à une princesse d’Allemagne, letter VIII
St-Pétersbourg 1768
Opera III 11, p.22-23

Theory and experiment

At this point I close my reflexions, since it seems to me I have sufficiently pondered the subject proposed as a problem and satisfactorily solved it. I did not think it necessary to confirm my theory by experiment, since it is entirely deduced from the most certain and unchallengeable principles of Mechanics, and therefore there can be no doubt at all whether it is true and can be applied in practice.

[remarkably self-confident for a not yet twenty-year-old student from Switzerland who has never seen a sea-going vessel!]

Hic tandem hisce meis meditationibus finem impono, cum uti videtur materiam in problemate propositam satis perpenderim, problematique satisfecerim. Haud opus esse existimavi istam meam theoriam experientia confirmare, cum integra & ex certissimis & irrepugnabilibus principiis Mechanicis deducta, atque adeo de illa dubitari, an vera sit ac an in praxi locum habere queat, minime possit.

E 4
Meditationes de Problemate Nautico de Implantatione Malorum
Prix Paris II (1727)
Opera II 20, p.35

Understanding nature by mathematical methods

From an early time, all the best mathematicians have been aware that the methods treated in this book are not only very useful in analysis itself, but can also contribute greatly to the solution of physical problems. For since the design of all the world has been most perfectly accomplished by the omniscient Creator, nothing at all happens in the world in which no reason involving some maximum or minimum can be seen. Hence there can be no doubt that all the effects in the world can be equally well determined, by way of the method of maxima and minima, from their final causes as from their efficient causes. Thus for example the curvature of a hanging rope or chain has been ascertained both ways, *a priori* from the impulses of gravity and also by the method of maxima and minima, since it is understood that the rope must assume the shape by which the centre of gravity obtains its lowest position. Similarly the curvature of rays penetrating a transparent medium of variable density has been determined both *a priori* and from the principle that they must reach a given point in the shortest time possible.

Iam pridem summi quique Geometrae agnoverunt Methodi in hoc Libro traditae non solum maximum esse usum in ipsa Analysi, sed etiam eam ad resolutionem Problematum physicorum amplissimum subsidium afferre. Cum enim Mundi universi fabrica sit perfectissima atque a Creatore sapientissimo absoluta, nihil omnino in mundo contingit, in quo non maximi minimive ratio quaepiam eluceat; quamobrem dubium prorsus est nullum, quin omnes Mundi effectus ex causis finalibus ope Methodi maximorum et minimorum aeque feliciter determinari queant, atque ex ipsis causis efficientibus. ... Hoc modo curvatura funis seu catenae suspensae duplici via est eruta, altera a priori ex sollicitationibus gravitatis, altera vero per Methodum maximorum ac minimorum, quoniam funis eiusmodi curvaturam recipere debere intelligebatur, cuius centrum gravitatis infimum obtineret locum. Similiter curvatura radiorum per medium diaphanum variae densitatis transeuntium tam a priori est determinata, quam etiam ex hoc principio, quod tempore brevissimo ad datum locum pervenire debeant.

E 65
Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes ...,
Additamentum I
Lausannae et Genevae 1744
Opera I 24, p.231

Models and their implementation - a thought experiment

Recently the following question arose on occasion of a permanent bridge that is to be constructed on the Neva river. Several persons had attempted to approach this task and had built models; the bridge itself was to be constructed by scaling these models up, and most people supposed that the bridge would be strong enough if the model just possessed a certain degree of strength. Indeed they thought, if only the model could carry a load proportional to the one the bridge would have to sustain, then without doubt the bridge itself – constructed in scale with the model – should have sufficient strength. However, it becomes quite evident that this conclusion is fallacious if one just considers the fact that such a bridge can certainly not be extended to an arbitrarily great length – say of one or several miles – without collapsing by its own weight, whatever strength the model may have had. It is therefore obvious that the strength of the bridge can on no account be determined from the strength of the model by the principle of similarity.

Orta est nuper haec quaestio occasione pontis perennis trans fluvium Nevam construendi. Cum enim plures hoc opus aggredi fuissent conati, atque in hunc finem modulos confecerint, ad quorum similitudinem ipse pons exstrui posset, plerique sunt arbitrati, pontem satis firmitatis esse habiturum, si modo modulus certo firmitatis gradu fuisset praeditus. Putarunt scilicet, si modo modulus simile onus gestare valeret, quale ipse pons sustinere debeat, tum nullum esse dubium, quin ipse pons secundum similitudinem moduli exstructus satis roboris esset habiturus. Hanc autem conclusionem esse fallacem, exinde satis est manifestum, quod talis pons certe non ad quantumvis magnam distantiam, veluti unius pluriumve milliarium extendi queat, quin proprio pondere corruat, quantumvis etiam roboris modulus habuisse videatur. Ex quo perspicuum est, firmitatem pontis neutiquam ex firmitate moduli secundum principium similitudinis definiri posse.

E 480
Regula facilis pro diiudicanda firmitate pontis ...
Novi Commentarii Academiae Scientiarum Petropolitanae 20 (1775)
Opera II 17, p.220

Rules of calculation and their explanation

Memorizing the rules of calculation without understanding does neither suffice to resolve all cases that may occur, nor does it sharpen the intellect, which should be the principal goal of teaching. We have therefore taken pains to explain the reason of all rules and operations in this manual in such a way that even people not yet proficient in profound studies can understand und accept them ... By this approach we hope to obtain the benefit that youths should – besides learning the proper skill in calculating – always bear the true reason of every operation in mind and thus get gradually used to sound reasoning. ... For every human being grasps and keeps in mind those facts much more easily of which he clearly sees the reason and origin, and is also able to make far better use of them in all cases that may occur.

Da nun die Erlernung der Rechenkunst ohne einigen Grund weder hinreichend ist, alle vorkommenden Fälle aufzulösen, noch den Verstand schärfet, als dahin die Absicht insonderheit gehen sollte: so hat man sich bemühet, in gegenwärtiger Anleitung von allen Regeln und Operationen den Grund so vorzutragen und zu erklären, dass denselben auch solche Leute, welche in gründlichen Abhandlungen noch nicht geübet sind, einsehen und verstehen ... Durch diese Einrichtung verhofft man also diesen Vortheil zu erlangen, dass die Jugend ausser der gehörigen Fertigkeit im Rechnen den wahren Grund von einer jeglichen Operation immer vor Augen habe, und dadurch zu gründlichem Nachdenken nach und nach angewöhnet werde. ... Dann ein jeglicher Mensch begreift und behält dasjenige im Gedächtnis viel leichter, wovon er den Grund und Ursprung deutlich einsieht, und weiss sich auch dasselbe bei allen vorkommenden Fällen weit besser zu Nutz zu machen.

E 17
Einleitung zur Rechen-Kunst zum Gebrauch des Gymnasii, St. Petersburg 1738
Opera III 2, p.3-4

Representing the Earth on maps

So one had to think of another manner of projection, that should firstly represent all meridians by straight lines and by which every degree of latitude should obtain the same size; but then again all parallels should intersect the meridians orthogonally. However, in this way it is quite impossible to have all the degrees on the parallels keeping the right ratio – viz. the one that can be observed on the spherical surface – to the degrees of the meridians; hence it seemed reasonable rather to stray somewhat from this ratio than to renounce the advantages mentioned above. Thus the following very important question arose: How should the meridians and parallels be assembled in such a way that the deviation from the true ratio which the degrees of longitude and latitude observe among themselves on the sphere is as small as possible throughout the extension of the map, so that the errors are barely perceptible? Indeed, such a deviation can easily be condoned if only the mentioned advantages are kept up.

De alia igitur proiectionis ratione erat cogitandum, quae primo omnes Meridianos per lineas rectas exhiberet, in quibus etiam omnes gradus latitudinis eandem quantitatem obtinerent; tum vero, ut omnes Paralleli Meridianos ad angulos rectos traiicerent. Quoniam vero hoc modo neutiquam fieri potest, ut ubique gradus Parallelorum ad gradus Meridianorum iustam teneant rationem, quae scilicet in superficie sphaerica deprehenditur, consultum visum est ab ista ratione potius aliquantillum aberrare, quam memoratis commodis renunciare. Hinc igitur sequens quaestio maximi momenti est nata: quomodo Meridiani cum Parallelis constitui debeant, ut a vera ratione, quam gradus longitudinis et latitudinis in Sphaera inter se tenent, per totam mappae extensionem quam minime aberretur? ita scilicet, ut errores vix percipi possent, quandoquidem talis aberratio facile condonari poterit, si modo memorata commoda obtineantur.

E 492
De Proiectione Geographica Delisliana in Mappa Generali Imperii Russici usitata
Acta Academiae Scientiarum Petropolitanae 1777/I (1778)
Opera I 28, p.289

Observation in pure mathematics

Among those particular properties of numbers that have been discovered and proved, most were indubitably at first just observed and noted by their inventors through the study of many individual numbers, before they thought about proving them. Thus in the series of those prime numbers which exceed some multiple of 4 by 1 – such as 5, 13, 17, 29, 37, 41 etc. –, it was no doubt much earlier observed that each one can be split into two squares than people started working on establishing the truth of this observation by a solid proof. ... We can rightly learn from this that in investigating the nature of numbers we should put much trust in the power of observation and induction, to which we owe the discovery of all those most elegant properties, and should therefore never desist from pursuing this work further even today.

Inter tot insignes numerorum proprietates, quae adhuc sunt inventae ac demonstratae, nullum est dubium, quin pleraeque primum ab inventoribus tantum sunt observatae et in multiplici numerorum tractatione animadversae, antequam de iis demonstrandis cogitaverint. Ita de eo numerorum primorum ordine, qui unitate superant multiplum quaternarii, cuiusmodi sunt 5, 13, 17, 29, 37, 41 etc., ante sine dubio est observatum eorum singulos in duo quadrata secari posse, quam in eo elaboratum, ut huius observationis veritas per solidam demonstrationem evinceretur. ... Ex quibus merito colligimus in numerorum indole scrutanda observationi et inductioni, cui omnes has elegantissimas proprietates acceptas referre debemus, plurimum esse tribuendum ideoque ne nunc quidem ab hoc negotio ulterius prosequendo esse desistendum.

E 256
Specimen de usu observationum in Mathesi pura
Novi Commentarii Academiae Scientiarum Petropolitanae 6, 1756/57 (1761)
Opera I 2, p.460-461

Fermat's "Last Theorem"

In Fermat there is another very beautiful theorem for which he claims to have found a proof. Indeed, on occasion of the Diophantine problem of finding two squares whose sum is a square, he states it is impossible to find two cubes whose sum is a cube, or two fourth powers whose sum is a fourth power, and in general that the formula $a^n + b^n = c^n$ is impossible whenever $n > 2$. Now I actually have found proofs that $a^3 + b^3 \neq c^3$ and $a^4 + b^4 \neq c^4$, where $\neq$ signifies that equality is impossible; but the proofs for these two cases are so different from one another that I see no chance of deriving from them a general proof of $a^n + b^n \neq c^n$ when $n > 2$. On the other hand, it can be seen – as it were through a veil but fairly plainly –, that the greater n, the more impossible the formula must be. Meanwhile I have not even yet been able to prove that the sum of two fifth powers cannot be a fifth power. To all appearances this proof rests on a fortunate inspiration, and as long as one does not chance upon this, all pondering may well be useless.

Bey *Fermat* findet sich noch ein sehr schönes *Theorema*, dessen *Demonstration* er sagt gefunden zu haben: Nehmlich bey Anlaß der *Diophantaeischen* Aufgabe zwey *Quadrata* zu finden deren *summ* ein *Quadrat* ist, sagt er daß es unmöglich sey zwey *cubos* zu finden deren *summ* ein *cubus* sey, und zwey *Biquadrata*, deren *summ* ein *Biquadratum*, und *generaliter* daß diese *Formul* $a^n + b^n = c^n$ allzeit unmöglich sey, wann $n > 2$. Ich habe nun wohl *Demonstrationen* gefunden daß $a^3 + b^3 \neq c^3$ und $a^4 + b^4 \neq c^4$, wo $\neq$ unmöglich gleich bedeutet; aber die *Demonstrationen* für diese zwey *casus* sind so von einander unterschieden, daß ich keine Möglichkeit sehe daraus eine allgemeine *Demonstration* für $a^n + b^n \neq c^n$ *si* $n > 2$ herzuleiten. Doch sieht man *quasi per transennam* ziemlich deutlich daß je grösser n ist, je unmöglicher die Formul seyn müsse; inzwischen habe ich noch nicht einmal beweisen können, daß *summa duarum potestatum quintarum* keine *potestas quinta* seyn könne. Dieser Beweiß beruhet allem Ansehen nach nur auf einem glücklichen Einfall, und so lang man nicht darauf verfällt, möchte wohl alles Nachsinnen vergebens seyn.

R 883
Leonhard Euler to Christian Goldbach, 4.8.1753
to appear in Opera IVA4