Euler's geometry

Euler's Characteristic
Is there a common feature between all these objects, and many others? Euler found the answer: V - E + F = 2, called Euler's formula where V = number of vertices, E = number of edges and F = number of faces.

Suppose any polyhedron is a rubber balloon. You can then flatten it by getting rid of one face and enlarging it around all others. For example, the cube becomes:

You now need to prove that in the new flattened balloon V - E + F = 1, since you got rid of one face, but of no vertex and of no edge.

Clearly, for any polygon in the plane, with n vertices, V = n, E = n, F = 1, and V - E + F = n - n + 1 = 1.

If one adds a polygon of x vertices and sharing y with another polygon, or with a group of polygons where V1 - E1 + F1 = 1, V = V1 + V2 + x - y, E = E1 + E2 + x - y + 1 and F = F1+1, therefore V - E + F = (V1 + V2 + x - y) - (E1 + E2 + x - y + 1) + (F1 + 1) = (V1 - E1 + F1) + (V2 - E2 + 1) - 1 = 1 + 1 - 1 = 1.

Thus V - E + F = 2 for any polyhedron (genus-zero).

The general Euler's Characteristic $\chi = V - E + F$ applies to non-convex polyhedra and even to their shapes:

- Torus: $\chi = 0$
- Double Torus: $\chi = -2$
- Klein bottle: $\chi = 0$
- Mobius Strip in Technorama, the Swiss Science Center: $\chi = 0$
- Edelweiss (stellated dodecahedron): $\chi = 4$

Euler's line
Leonhard Euler showed that in any triangle, the orthocenter (blue), the circumcenter (green), the centroid (yellow), and the center of the nine-point circle (red points) are collinear. This line is called Euler's line.

Euler's brick
An Euler brick is a cuboid with integer edges and integer face diagonals. A primitive Euler brick is an Euler brick with its edges relatively prime or equivalently a solution to the following system of diophantine equations:

\[ a^2 + b^2 = c^2 \]
\[ b^2 + c^2 = d^2 \]
\[ a^2 + c^2 = f^2 \]

Euler found at least two parametric solutions to the problem. The smallest Euler brick has edges (a,b,c) = (240,117,44) and faces diagonals 267, 244, and 125. Paul Halcke discovered it in 1719.

Other solutions are (275, 252, 240), (693, 480, 140), (720, 132, 85), (792, 231, 160).

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